University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

Exercises 26-30 provide experience relevant to the motion of relativistic charged particles in EM fields.

26.
The field strength and dual field strength tensors are

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} ,$$

where $\epsilon^{\mu\nu\rho\sigma} = 1$ (-1) when $\mu\nu\rho\sigma$ are even (odd) permutations of 0123, and 0 otherwise. You may use their explicit elements as given in SCSR Eqs. (62) and (65).
(a.)
By explicit evaluation, show that $F^{\mu\nu} F_{\mu\nu}$ is proportional to $E^2 - c^2 B^2$, and find the constant of proportionality. (Because $F^{\mu\nu} F_{\mu\nu}$ is obviously a Lorentz scalar, the Lorentz invariance of $E^2 - c^2 B^2$ is therefore said to be *manifest*.)
(b.)
By explicit evaluation, show that $F^{\mu\nu} G_{\mu\nu}$ is proportional to $\vec{E} \cdot \vec{B}$, and find the constant of proportionality. (Likewise the Lorentz invariance of $\vec{E} \cdot \vec{B}$ is manifest.)
(c.)
What two criteria must (uniform nonzero) $\vec{E}$ and $\vec{B}$ satisfy in the lab frame so that, in a different inertial frame, $\vec{B}$ is allowed to vanish?

27.
Griffiths Problem 12.36.

28.
You are an indefagitable runner of rest mass m, whose feet generate a constant force F_0 in the x direction (as observed in the *laboratory*). This force causes your (relativistic) momentum to increase linearly with laboratory time t.
(a.)
At $t = 0$, when you are at rest at the origin, your feet begin to exert this force. Thereafter, show that $\sinh\eta$, where η is your rapidity, is proportional to t, and find the constant of proportionality.
(b.)
At $t = t_1$, a laser pulse is shot from the origin in the x direction. How much of a head start t_1 do you require in order for the laser pulse never to catch up with you?

29.
At $t = 0$, a particle of rest mass m and charge e is at rest at the origin. It accelerates under the influence of a uniform static electric field $\vec{E} = \hat{z} E_0$.
(a.)
For $t > 0$, show that the relativistic solution for $z(t)$ is given by

$$z(t) = \int_0^t c\beta_z(t)\, dt$$
$$\beta_z(t) = \tanh\eta(t)$$
$$\eta(t) = \sinh^{-1} \frac{eE_0}{mc} t .$$

(b.)
Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x} p_\perp$. Show that the solution for $z(t)$ is the same as in (a.), except that m is replaced by m_{eff}, where

$$m_{\text{eff}} = \sqrt{m^2 + p_\perp^2/c^2} .$$

This says that, under the influence of a uniform electrostatic field, the longitudinal motion of a particle with nonzero transverse momentum is the same as that of a heavier particle with zero transverse momentum.
(c.)
Under the conditions of part (b.), for $t > 0$ does $x(t)$ increase linearly with t? Explain.

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30.

At $t = 0$, a particle of rest mass m and charge e is at the origin, with initial momentum $\vec{p}(0) = \hat{x}p_\perp$. It accelerates under the influence of a uniform static magnetic field $\vec{B} = \hat{z}B_0$.

(a.)

For $t > 0$, show that the relativistic solution for $x(t)$ and $y(t)$ is given by

$$x(t) = \int_0^t c\beta_x(t)\, dt$$

$$y(t) = \int_0^t c\beta_y(t)\, dt$$

$$\gamma_\perp mc\beta_x(t) = p_\perp \cos\omega_c t$$

$$\gamma_\perp mc\beta_y(t) = p_\perp \sin\omega_c t$$

$$\omega_c = -\frac{eB_0}{\gamma_\perp m}$$

$$\gamma_\perp m = \sqrt{m^2 + p_\perp^2/c^2} \; .$$

(b.)

Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x}p_\perp + \hat{z}p_\parallel$. Show that the solution for $x(t)$ and $y(t)$ is the same as in (a.), except that $\gamma_\perp$ is replaced by γ, where

$$\gamma m = \sqrt{m^2 + p_\perp^2/c^2 + p_\parallel^2/c^2} \; .$$

(c.)

Show that the result of (b.) alternatively can be expressed as the result of (a.) with m replaced by

$$m_{\text{eff}} = \sqrt{m^2 + p_\parallel^2/c^2} \; .$$

This says that, under the influence of a uniform magnetostatic field, the transverse motion of a particle with nonzero longitudinal momentum is the same as that of a heavier particle with zero longitudinal momentum.

(d.)

Under the conditions of part (b.), for $t > 0$ does $z(t)$ increase linearly with t? Explain.